

## Differentialrechnung

Funktionen

$$\mathbf{f} : (D \subseteq \mathbb{R}^n) \rightarrow \mathbb{R}^m \quad \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \rightarrow \mathbf{f}(\mathbf{x}) = \begin{pmatrix} f_1(\mathbf{x}) \\ \vdots \\ f_m(\mathbf{x}) \end{pmatrix} \quad i \in [1 \dots m] \quad j \in [1 \dots n]$$

$$\text{Jacobi-Matrix} \quad \mathbf{J}(\mathbf{f}(\mathbf{x})) = \frac{\partial \mathbf{f}}{\partial \mathbf{x}^T}(\mathbf{x}) = (j_{ij}(\mathbf{x})) = \left( \frac{\partial f_i}{\partial x_j}(\mathbf{x}) \right) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{x}) & \dots & \frac{\partial f_1}{\partial x_n}(\mathbf{x}) \\ \dots & \frac{\partial f_i}{\partial x_j}(\mathbf{x}) & \dots \\ \frac{\partial f_m}{\partial x_1}(\mathbf{x}) & \dots & \frac{\partial f_m}{\partial x_n}(\mathbf{x}) \end{pmatrix}$$

$$\text{Grenzwert} \quad \lim(\mathbf{f}(\mathbf{x})) = \sum_{i=1}^m \lim(f_i(\mathbf{x})) \mathbf{e}_i$$

$$\text{Ableitung} \quad \frac{\partial \mathbf{f}}{\partial x_j}(\mathbf{x}) = \sum_{i=1}^m j_{ij} \mathbf{e}_i$$

$$\text{Differential} \quad d\mathbf{f}(\mathbf{x}) = \sum_{i=1}^m \sum_{j=1}^n j_{ij} dx_j \mathbf{e}_i \quad d^k \mathbf{f}(\mathbf{x}) = \sum_{i=1}^m \left( \sum_{j=1}^n dx_j \frac{\partial}{\partial x_j} \right)^k f_i(\mathbf{x}) \mathbf{e}_i$$

Produktregel      Quotientenregel      Kettenregel      Partielle Integration

$$\frac{d}{dt}(uv) = \dot{u}v + u\dot{v} \quad \frac{d}{dt}\left(\frac{u}{v}\right) = \frac{\dot{u}v - u\dot{v}}{v^2} \quad \frac{d}{dt}(u(v)) = \frac{du}{dv} \frac{dv}{dt} \quad \int_a^b uv \, dt = [uv]_a^b - \int_a^b u \dot{v} \, dt$$

**Nabla**

$$\text{Gradient} \quad \text{grad}(f_i(\mathbf{x})) = \nabla f_i = \sum_{j=1}^n j_{ij} \mathbf{e}_j = \lim_{V \rightarrow 0} \left( \frac{1}{V} \oint_{\mathbf{A}=\delta V} f_i d\mathbf{A} \right)$$

$m = n$ :

$$\text{Divergenz} \quad \text{div}(\mathbf{f}(\mathbf{x})) = \nabla \cdot \mathbf{f} = \sum_{i=1}^m j_{ii} = \lim_{V \rightarrow 0} \left( \frac{1}{V} \oint_{\mathbf{A}=\delta V} \mathbf{f} \cdot d\mathbf{A} \right)$$

$$\text{Laplace} \quad \Delta \mathbf{f}(\mathbf{x}) = \nabla^2 \mathbf{f} = \sum_{i=1}^m \sum_{j=1}^n \frac{\partial j_{ij}}{\partial x_j} \mathbf{e}_i$$

$m = n = 3$ :

$$\text{Rotation} \quad \text{rot}(\mathbf{f}(\mathbf{x})) = \nabla \times \mathbf{f} = \begin{pmatrix} j_{32} - j_{23} \\ j_{13} - j_{31} \\ j_{21} - j_{12} \end{pmatrix} = \lim_{V \rightarrow 0} \left( \frac{1}{V} \oint_{\mathbf{A}=\delta V} d\mathbf{A} \times \mathbf{f} \right) = \frac{\mathbf{A}}{|\mathbf{A}|} \lim_{\mathbf{A} \rightarrow \mathbf{0}} \left( \frac{1}{|\mathbf{A}|} \oint_{\mathbf{s}=\delta \mathbf{A}} \mathbf{f} \cdot d\mathbf{s} \right)$$

$$\text{Identitäten} \quad \nabla \times (\nabla f) = \mathbf{0} \quad \nabla \cdot (\nabla \times \mathbf{f}) = 0$$

## Integralsätze

Differenz von Funktionswerten      Integral Vektorfeld über Kurve      Fluß Vektorfeld durch Fläche      Volumintegral

$$f(\mathbf{b}) - f(\mathbf{a})$$

$$\int_s \mathbf{v}(\mathbf{x}) \cdot d\mathbf{x}$$

$$\iint_{\mathbf{A}} \mathbf{w}(\mathbf{x}) \cdot d\mathbf{A}$$

$$\iiint_V g(\mathbf{x}) dV$$

Hauptsatz

Stokes

Gauß

$$f(\mathbf{b}) - f(\mathbf{a}) = \int_{\mathbf{s}=\mathbf{a}}^{\mathbf{b}} \mathbf{v}(\mathbf{x}) \cdot d\mathbf{x}$$

$$\oint_{\mathbf{s}=\delta \mathbf{A}} \mathbf{v}(\mathbf{x}) \cdot d\mathbf{x} = \iint_{\mathbf{A}} \mathbf{w}(\mathbf{x}) \cdot d\mathbf{A}$$

$$\oint_{\mathbf{A}=\delta V} \mathbf{w}(\mathbf{x}) \cdot d\mathbf{A} = \iiint_V g(\mathbf{x}) dV$$

Bedingung

Bedingung

Bedingung

$$\nabla \times \mathbf{v} = \mathbf{0} \quad \Leftrightarrow \quad \mathbf{v} = \nabla f$$

$$\nabla \cdot \mathbf{w} = 0 \quad \Leftrightarrow \quad \mathbf{w} = \nabla \times \mathbf{v}$$

$$g = \nabla \cdot \mathbf{w}$$

## Koordinatensysteme

|                            | <b>Kartesisch</b>   | <b>Zylinder</b>   | <b>Kugel</b>  |
|----------------------------|---|---|---|
| <b>Vektoren</b>            | $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\infty \dots \infty \\ -\infty \dots \infty \\ -\infty \dots \infty \end{pmatrix}$ $\begin{pmatrix} \sqrt{x^2 + y^2} \\ \arctan\left(\frac{y}{x}\right) \\ z \end{pmatrix}$ $\begin{pmatrix} \sqrt{x^2 + y^2 + z^2} \\ \arctan\left(\frac{y}{x}\right) \\ \arctan\left(\frac{\sqrt{x^2 + y^2}}{z}\right) \end{pmatrix}$   | $\begin{pmatrix} \rho \cos(\varphi) \\ \rho \sin(\varphi) \\ z \end{pmatrix}$ $\begin{pmatrix} \rho \\ \varphi \\ z \end{pmatrix} = \begin{pmatrix} 0 \dots \infty \\ 0 \dots 2\pi \\ -\infty \dots \infty \end{pmatrix}$ $\begin{pmatrix} \sqrt{\rho^2 + z^2} \\ \varphi \\ \arctan\left(\frac{\rho}{z}\right) \end{pmatrix}$  | $\begin{pmatrix} r \cos(\varphi) \sin(\vartheta) \\ r \sin(\varphi) \sin(\vartheta) \\ r \cos(\vartheta) \end{pmatrix}$ $\begin{pmatrix} r \sin(\vartheta) \\ \varphi \\ r \cos(\vartheta) \end{pmatrix}$ $\begin{pmatrix} r \\ \varphi \\ \vartheta \end{pmatrix} = \begin{pmatrix} 0 \dots \infty \\ 0 \dots 2\pi \\ 0 \dots \pi \end{pmatrix}$   |
| <b>Wegelement ds</b>       | $\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}$  | $\begin{pmatrix} d\rho \\ \rho d\varphi \\ dz \end{pmatrix}$  | $\begin{pmatrix} dr \\ r \sin(\vartheta) d\varphi \\ r d\vartheta \end{pmatrix}$  |
| <b>Einheitsvektoren</b>    | $\begin{pmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\begin{pmatrix} \cos(\varphi) & \sin(\varphi) & 0 \\ -\sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\begin{pmatrix} \cos(\varphi) \sin(\vartheta) & \sin(\varphi) \sin(\vartheta) & \cos(\vartheta) \\ -\sin(\varphi) & \cos(\varphi) & 0 \\ \cos(\varphi) \cos(\vartheta) & \sin(\varphi) \cos(\vartheta) & -\sin(\vartheta) \end{pmatrix}$ | $\begin{pmatrix} \cos(\varphi) & -\sin(\varphi) & 0 \\ \sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\begin{pmatrix} \mathbf{e}_\rho & \mathbf{e}_\varphi & \mathbf{e}_z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\begin{pmatrix} \sin(\vartheta) & 0 & \cos(\vartheta) \\ 0 & 1 & 0 \\ \cos(\vartheta) & 0 & -\sin(\vartheta) \end{pmatrix}$ | $\begin{pmatrix} \cos(\varphi) \sin(\vartheta) & -\sin(\varphi) \cos(\varphi) \cos(\vartheta) \\ \sin(\varphi) \sin(\vartheta) & \cos(\varphi) \sin(\varphi) \cos(\vartheta) \\ \cos(\vartheta) & 0 & -\sin(\vartheta) \end{pmatrix}$ $\begin{pmatrix} \sin(\vartheta) & 0 & \cos(\vartheta) \\ 0 & 1 & 0 \\ \cos(\vartheta) & 0 & -\sin(\vartheta) \end{pmatrix}$ $\begin{pmatrix} \mathbf{e}_r & \mathbf{e}_\varphi & \mathbf{e}_\vartheta \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ |
| <b>Nabla</b>               |   |   |   |
| $\nabla f$                 | $\begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix}$   | $\begin{pmatrix} \frac{\partial f}{\partial \rho} \\ \frac{1}{\rho} \frac{\partial f}{\partial \varphi} \\ \frac{\partial f}{\partial z} \end{pmatrix}$   | $\begin{pmatrix} \frac{\partial f}{\partial r} \\ \frac{1}{r \sin(\vartheta)} \frac{\partial f}{\partial \varphi} \\ \frac{1}{r} \frac{\partial f}{\partial \vartheta} \end{pmatrix}$   |
| $\nabla \cdot \mathbf{f}$  | $\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z}$   | $\frac{1}{\rho} \frac{\partial(\rho f_\rho)}{\partial \rho} + \frac{1}{\rho} \frac{\partial f_\varphi}{\partial \varphi} + \frac{\partial f_z}{\partial z}$   | $\frac{1}{r^2} \frac{\partial(r^2 f_r)}{\partial r} + \frac{1}{r \sin(\vartheta)} \frac{\partial f_\varphi}{\partial \varphi} + \frac{1}{r \sin(\vartheta)} \frac{\partial(\sin(\vartheta) f_\vartheta)}{\partial \vartheta}$   |
| $\nabla \times \mathbf{f}$ | $\begin{pmatrix} \frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z} \\ \frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x} \\ \frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y} \end{pmatrix}$   | $\begin{pmatrix} \frac{1}{\rho} \frac{\partial f_z}{\partial \varphi} - \frac{\partial f_\varphi}{\partial z} \\ \frac{\partial f_\rho}{\partial z} - \frac{\partial f_z}{\partial \rho} \\ \frac{1}{\rho} \left( \frac{\partial(\rho f_\varphi)}{\partial \rho} - \frac{\partial f_\rho}{\partial \varphi} \right) \end{pmatrix}$  | $\begin{pmatrix} \frac{1}{r \sin(\vartheta)} \left( \frac{\partial(\sin(\vartheta) f_\varphi)}{\partial \vartheta} - \frac{\partial f_\vartheta}{\partial \varphi} \right) \\ \frac{1}{r} \left( \frac{\partial(r f_\vartheta)}{\partial r} - \frac{\partial f_r}{\partial \vartheta} \right) \\ \frac{1}{r} \left( \frac{1}{\sin(\vartheta)} \frac{\partial f_r}{\partial \varphi} - \frac{\partial(r f_\varphi)}{\partial r} \right) \end{pmatrix}$   |
| $\nabla^2 \mathbf{f}$      | $\frac{\partial^2 \mathbf{f}}{\partial x^2} + \frac{\partial^2 \mathbf{f}}{\partial y^2} + \frac{\partial^2 \mathbf{f}}{\partial z^2}$  | $\frac{\partial^2 \mathbf{f}}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial \mathbf{f}}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 \mathbf{f}}{\partial \varphi^2} + \frac{\partial^2 \mathbf{f}}{\partial z^2}$   | $\frac{\partial^2 \mathbf{f}}{\partial r^2} + \frac{2}{r} \frac{\partial \mathbf{f}}{\partial r} + \frac{1}{r^2 \sin(\vartheta)^2} \frac{\partial^2 \mathbf{f}}{\partial \varphi^2} + \frac{1}{r^2 \sin(\vartheta)} \frac{\partial(\sin(\vartheta) \frac{\partial \mathbf{f}}{\partial \vartheta})}{\partial \vartheta}$  |

## Kurvendiskussion

Tangenteneinheitsvektor    Hauptnormalenvektor

$$\mathbf{T}(x_0) = \frac{\frac{d\mathbf{f}}{dx}(x_0)}{\left|\frac{d\mathbf{f}}{dx}(x_0)\right|}$$

$$\mathbf{N}(x_0) = \frac{\frac{d\mathbf{T}}{dx}(x_0)}{\left|\frac{d\mathbf{T}}{dx}(x_0)\right|}$$

Binormalenvektor

$$\mathbf{B}(x_0) = \mathbf{T}(x_0) \times \mathbf{N}(x_0)$$

Krümmung

$$\kappa(x) = \frac{|\frac{d\mathbf{T}}{ds}|}{ds}$$

Torsion

$$\tau(x) = -\mathbf{N} \frac{d\mathbf{B}}{ds}$$

Bogenlänge

$$\left(\frac{ds}{dx}\right)^2 = \sum_{i=1}^m \left(\frac{df_i}{dx}(x)\right)^2$$

Bahngeschwindigkeit

$$\frac{d\mathbf{f}}{dx}(x) = \frac{ds(x)}{dx} \mathbf{T}(x)$$

Beschleunigung

$$\frac{d^2\mathbf{f}}{dx^2}(x) = \frac{d^2s}{dx^2}(x) \mathbf{T}(x) + \left(\frac{ds}{dx}(x)\right)^2 \kappa(x) \mathbf{N}(x)$$

Tangente

$$\mathbf{t}(x_0, \lambda) = \mathbf{f}(x_0) + \lambda \mathbf{T}(x_0)$$

Normale

$$\mathbf{n}(x_0, \lambda) = \mathbf{f}(x_0) + \lambda \mathbf{N}(x_0)$$

Hessematrix (symmetrisch)

$$\mathbf{H}(f(\mathbf{x})) = \begin{pmatrix} f_{x_1 x_1} & \dots & f_{x_1 x_n} \\ \dots & \dots & \dots \\ f_{x_n x_1} & \dots & f_{x_n x_n} \end{pmatrix}$$

Richtungsableitung

$$\frac{\partial f}{\partial \mathbf{a}}(\mathbf{x}) = \nabla f(\mathbf{x}) \cdot \frac{\mathbf{a}}{|\mathbf{a}|}$$

Tangentialebene

$$(\mathbf{x} - \mathbf{x}_0) \cdot \nabla f(\mathbf{x}_0) = 0$$

Extremum

$$\Rightarrow df(\mathbf{x}_0) = 0 \quad \Leftrightarrow \quad \nabla f(\mathbf{x}_0) = \mathbf{0} \quad \mathbf{H}(f(\mathbf{x})) \text{ positiv (negativ) definit} \Rightarrow \text{Minimum (Maximum)}$$

$$\text{indefinit} \Rightarrow \text{Sattelpunkt}$$

$$\text{sonst} \Rightarrow \text{keine Aussage}$$

## Fehlerrechnung

$$|\Delta f(\mathbf{x})| \leq \sum_j \left| \frac{\partial f}{\partial x_j}(\mathbf{x}) \right| \Delta x_j \quad \begin{array}{l} x = \text{Mittelwerte} \\ \Delta x_j = \text{maximaler absoluter Fehler} \end{array}$$

## Integrale

Parameterabhängige Integrale (Leibnizsche Regel)

$$\frac{d}{d\lambda} \int_{a(\lambda)}^{b(\lambda)} f(x, \lambda) dx = \int_{a(\lambda)}^{b(\lambda)} \frac{df}{d\lambda}(x, \lambda) dx + f(b(\lambda), \lambda) \frac{db}{d\lambda}(\lambda) - f(a(\lambda), \lambda) \frac{da}{d\lambda}(\lambda)$$

Kurvenintegral 1. Art  $\Leftrightarrow$  2. Art

$$\int_S f(\mathbf{x}) dx = \int_s f(\mathbf{x}(t)) |\dot{\mathbf{x}}(t)| dt = \int_s \left( f(\mathbf{x}(t)) \frac{\dot{\mathbf{x}}(t)}{|\dot{\mathbf{x}}(t)|} \right) \cdot (\dot{\mathbf{x}}(t) dt) = \int_s \mathbf{v}(\mathbf{x}(t)) \cdot \dot{\mathbf{x}}(t) dt = \int_S \mathbf{v}(\mathbf{x}) \cdot d\mathbf{x}$$

Flächenintegral 1. Art  $\Leftrightarrow$  2. Art

$$\iint_A f(\mathbf{x}) dA = \iint_a f(\mathbf{x}(s, t)) |\mathbf{x}_s \times \mathbf{x}_t| ds dt = \iint_a \left( f(\mathbf{x}(s, t)) \frac{\mathbf{x}_s \times \mathbf{x}_t}{|\mathbf{x}_s \times \mathbf{x}_t|} \right) \cdot ((\mathbf{x}_s \times \mathbf{x}_t) ds dt) = \iint_a \mathbf{v}(\mathbf{x}(s, t)) \cdot (\mathbf{x}_s \times \mathbf{x}_t) ds dt = \iint_A \mathbf{v}(\mathbf{x}) \cdot d\mathbf{A}$$

## Anwendungen

Rotationskörper     $A = 2\pi \int_0^s y ds = 2\pi y_s s$      $V = \pi \int_a^b y^2 dx = 2\pi y_s A$

Statisches Moment     $M = \sum_i m_i d_i$      $M_x = \int_0^s y ds = \frac{1}{2} \int_a^b y^2 dx$      $M_y = \int_0^s x ds = \int_a^b xy dx$

Schwerpunkt     $S = (x_s, y_s) = \left( \frac{1}{s} \int_0^s x ds, \frac{1}{s} \int_0^s y ds \right) = \left( \frac{1}{A} \int_a^b xy dx, \frac{1}{2A} \int_a^b y^2 dx \right)$

## Interpolation

Lagrange

$$P_n(x) = \sum_{i=1}^n y_i \prod_{j=1, j \neq i}^n \frac{x-x_j}{x_i-x_j}$$

Newton

$$P_n(x) = \sum_{i=1}^n \alpha_i \prod_{j=1}^i (x-x_j)$$

Trapezformel

$$\int_a^b f(x)dx = h \left( \frac{y_0+y_n}{2} + \sum_{i=1}^{n-1} y_i \right)$$

Simpsonsche Formel

$$\int_a^b f(x)dx = \frac{h}{3} \left( y_0 + y_n + 2 \sum_{i=1}^{\frac{n-1}{2}} y_{2i} + 4 \sum_{i=1}^{\frac{n-1}{2}} y_{2i+1} \right)$$

Schrittweises Integrieren  $y' = f(x, y)$

Euler-Verfahren

$$y_{k\tau_1} = y_k + hf(x_k, y_k)$$

Runge-Kutta-Verfahren

$$y_{k\tau_1} = y_k + \frac{h}{6} (T_{1k} + 2T_{2k} + 2T_{3k} + T_{4k})$$

$$T_{1k} = f(x_k, y_k) \quad T_{2k} = f\left(x_k + \frac{h}{2}, y_k + \frac{h}{2}T_{1k}\right)$$

$$T_{3k} = f\left(x_k + \frac{h}{2}, y_k + \frac{h}{2}T_{2k}\right) \quad T_{4k} = f(x_k + h, y_k + hT_{3k})$$

## Differentialgleichungen

$$y' = g(x)h(y) \quad y = y_0 \text{ mit } h(y_0) = 0 \Rightarrow \int \frac{1}{h(y)} dy = \int g(x) dx + c$$

$$y' = f\left(\frac{y}{x}\right) \quad u = \frac{y}{x} \Rightarrow \int \frac{1}{f(u)-u} du = \ln(|x|) + c$$

$$F(x, y', y'') = 0 \quad u = y' \Rightarrow F(x, u, u') = 0 \quad y = \int u(x, c_1) dx$$

$$F(y, y', y'') = 0 \quad u(y) = y' \Rightarrow F(y, u, u'u) = 0 \quad \int \frac{1}{u(y, c_1)} dy = x + c_2$$

$$y' + p(x)y = q(x)y^n \quad u = y^{1-n} \Rightarrow u' + (1-n)p(x)u = (1-n)q(x)$$

## Lineare Differentialgleichungen

$$\sum_{i=0}^n f_i y^{(i)} = p(x) = Q_\alpha(x) \sin(\nu x + \beta) \exp(\mu x)$$

$$y = \sum_{i=1}^n c_i y_i + y_p$$

Wronski-Determinante  $\neq 0 \Rightarrow y_1 \dots y_n$  Fundamentalsystem Inhomogen: Variation der Konstanten

$$W(y_1, y_2, \dots, y_n) = \begin{vmatrix} y_1 & \dots & y_n \\ \dots & \dots & \dots \\ y_1^{(n-1)} & \dots & y_n^{(n-1)} \end{vmatrix} \neq 0 \quad \begin{pmatrix} y_1 & \dots & y_n \\ \dots & \dots & \dots \\ y_1^{(n-1)} & \dots & y_n^{(n-1)} \end{pmatrix} \cdot \begin{pmatrix} c'_1 \\ \vdots \\ c'_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ p(x) \end{pmatrix}$$

$f_i = \text{const}$

$$\text{Charakteristisches Polynom} \quad P(\lambda) = \sum_{i=0}^n f_i \lambda^i = 0 = \prod_{j=0}^{m \leq n} (\lambda - \lambda_j)^{k_j}$$

$$y = \sum_{j=0}^m \sum_{i=0}^{k_j-1} c_{ji} x^i \exp(\lambda_j x) \quad y_p = R_\alpha(x) x^k \exp(\lambda x) = (S_\alpha(x) \sin(\nu x) + T_\alpha(x) \cos(\nu x)) x^k \exp(\mu x)$$