Matrizen

Matrix

$$\mathbf{A} = \left(a_{ij}\right) = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \cdots & a_{ij} & \cdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

inverse

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|} \begin{pmatrix} A_{11} & \cdots & A_{n1} \\ \cdots & A_{ij} & \cdots \\ A_{1m} & \cdots & A_{nm} \end{pmatrix}$$

Adjunkte

$$A_{ij} = (-1)^{i+j} \left| \begin{pmatrix} a_{11} & \cdots & a_{i(j-1)} & a_{1(j+1)} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{(i-1)1} & \cdots & a_{(i-1)(j-1)} & a_{(i-1)(j+1)} & \cdots & a_{(i-1)n} \\ a_{(i+1)1} & \cdots & a_{(i+1)(j-1)} & a_{(i+1)(j+1)} & \cdots & a_{(i+1)n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & \cdots & a_{m(j-1)} & a_{m(j+1)} & \cdots & a_{mn} \end{pmatrix} \right|$$

Determinante

$$\det(\mathbf{A}) = |\mathbf{A}| = \left|\mathbf{A}^{\mathrm{T}}\right| = \sum_{i=1}^{m} a_{ij} A_{ij} = \sum_{j=1}^{n} a_{ij} A_{ij}$$

Einheitsmatrix

$$\mathbf{1} = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix} = \mathbf{A} \cdot \mathbf{A}^{-1}$$

Transposition

$$\mathbf{A}^{\mathrm{T}} = \left(a_{ji}\right) \quad (\mathbf{A} \cdot \mathbf{B})^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}} \cdot \mathbf{A}^{\mathrm{T}} \qquad (\mathbf{A} \cdot \mathbf{B})^{-1} = \mathbf{B}^{-1} \cdot \mathbf{A}^{-1}$$

Rechenregeln

$$\left|\begin{pmatrix} (\mathbf{a}_1 + \mathbf{a}_2) & \mathbf{a}_3 \end{pmatrix}\right| = \left|\begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_3 \end{pmatrix}\right| + \left|\begin{pmatrix} \mathbf{a}_2 & \mathbf{a}_3 \end{pmatrix}\right| \qquad \lambda \left|\begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{pmatrix}\right| = \left|\begin{pmatrix} \lambda \mathbf{a}_1 & \mathbf{a}_2 \end{pmatrix}\right| \qquad |\mathbf{A} \cdot \mathbf{B}| = |\mathbf{A}|\,|\mathbf{B}|$$

$$\left|\begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{pmatrix}\right| = \left|\begin{pmatrix} \mathbf{a}_1 & (\mathbf{a}_2 + \lambda \mathbf{a}_1) \end{pmatrix}\right| \qquad \left|\begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{pmatrix}\right| = -\left|\begin{pmatrix} \mathbf{a}_2 & \mathbf{a}_1 & \mathbf{a}_3 \end{pmatrix}\right| \qquad \left|\begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_1 & \mathbf{a}_2 \end{pmatrix}\right| = 0$$

Eigenwerte λ_i

$$P(\lambda) = \det(\mathbf{A} - \lambda \mathbf{1}) = \prod_{i=0}^{k} (\lambda - \lambda_i)^{\alpha_i} = 0$$

Eigenvektoren $\boldsymbol{\lambda}_{ij}$ zu λ_i

$$(\mathbf{A} - \lambda_i \mathbf{1}) \cdot \boldsymbol{\lambda}_{ij} = \mathbf{0} \quad 0 \le j \le \beta_i$$

Vielfachheiten

$1 \le \beta_i \le \alpha_i$
α_i algebraische Vf.
β_i geometrische Vf.

Orthogonalisieren

$$\mathbf{b}_j = \boldsymbol{\lambda}_{ij} - \sum_{q=1}^{j-1} \frac{\boldsymbol{\lambda}_{ij} \cdot \mathbf{b}_q}{\mathbf{b}_q^2} \mathbf{b}_q \qquad \mathbf{B} = \left(\mathbf{b}_j\right)$$

Definitheit

$\forall \mathbf{x} \neq \mathbf{0} : \overline{\mathbf{x}}^{\mathrm{T}} \cdot \mathbf{A} \cdot \mathbf{x} > 0 \Leftrightarrow \forall i : \lambda_i > 0 \Leftrightarrow$ **A** ist positiv definit
$\forall \mathbf{x} \neq \mathbf{0} : \overline{\mathbf{x}}^{\mathrm{T}} \cdot \mathbf{A} \cdot \mathbf{x} \ge 0 \Leftrightarrow \forall i : \lambda_i \ge 0 \Leftrightarrow$ **A** ist positiv semidefinit
$\forall \mathbf{x} \neq \mathbf{0} : \overline{\mathbf{x}}^{\mathrm{T}} \cdot \mathbf{A} \cdot \mathbf{x} < 0 \Leftrightarrow \forall i : \lambda_i < 0 \Leftrightarrow$ **A** ist negativ definit
$\forall \mathbf{x} \neq \mathbf{0} : \overline{\mathbf{x}}^{\mathrm{T}} \cdot \mathbf{A} \cdot \mathbf{x} \le 0 \Leftrightarrow \forall i : \lambda_i \le 0 \Leftrightarrow$ **A** ist negativ semidefinit
sonst: **A** ist indefinit

Vektor

Spaltenvektor

$$\mathbf{a} = (a_i) = \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix}$$

Zeilenvektor

$$\mathbf{a}^{\mathrm{T}} = \left(a_j\right) = \begin{pmatrix} a_1 & \cdots & a_n \end{pmatrix}$$

Skalarprodukt

$$\mathbf{a}_1 \cdot \mathbf{a}_2 = \mathbf{a}_1^{\mathrm{T}} \cdot \mathbf{a}_2$$

Kreuzprodukt

$$\mathbf{a}_1 \times \mathbf{a}_2 = \begin{pmatrix} a_{12}a_{23} - a_{13}a_{22} \\ a_{13}a_{21} - a_{11}a_{23} \\ a_{11}a_{22} - a_{12}a_{21} \end{pmatrix}$$

Betrag

$$|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$$

Division

$$\frac{\mathbf{a}_1}{\mathbf{a}_2} = \frac{\mathbf{a}_1 \cdot \mathbf{a}_2}{\mathbf{a}_2^2}$$

elementweise

$$\frac{\mathbf{a}_1}{\mathbf{a}_2^{\mathrm{T}}} = \begin{pmatrix} a_{11} \\ \vdots \\ a_{1m} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{a_{21}} & \cdots & \frac{1}{a_{2n}} \end{pmatrix}$$

Richtungsableitung

$$\frac{\partial \mathbf{a}_1}{\partial \mathbf{a}_2} = (\mathbf{a}_2 \cdot \nabla)\, \mathbf{a}_1$$

Jacobi-Matrix

$$\frac{\partial \mathbf{a}_1}{\partial \mathbf{a}_2^{\mathrm{T}}} = \begin{pmatrix} \frac{\partial a_{11}}{\partial a_{21}} & \cdots & \frac{\partial a_{11}}{\partial a_{2n}} \\ \cdots & \frac{\partial a_{1i}}{\partial a_{2j}} & \cdots \\ \frac{\partial a_{1m}}{\partial a_{21}} & \cdots & \frac{\partial a_{1m}}{\partial a_{2n}} \end{pmatrix}$$

Gleichungssystem (Cramer)

$$\mathbf{A} \cdot \mathbf{x} = \mathbf{b} \quad x_i = \frac{\left|\begin{pmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_{i-1} & \mathbf{b} & \mathbf{a}_{i+1} & \cdots & \mathbf{a}_n \end{pmatrix}\right|}{|\mathbf{A}|}$$

Quadriken

$$a(\mathbf{x}) = \left(\sum_{i=1}^{n} a_i x_i\right)^2 + \sum_{i=1}^{n} d_i x_i + c = \left(\mathbf{x}^T \cdot \mathbf{A} \cdot \mathbf{x}\right) + \left(\mathbf{d}^T \cdot \mathbf{x}\right) + c = \sum_{i=1}^{n} (\lambda_i z_i)^2 + c' = 0$$

$$a(\mathbf{x}) = \left(\mathbf{y}^T \cdot \left(\mathbf{B}^T \cdot \mathbf{A} \cdot \mathbf{B}\right) \cdot \mathbf{y}\right) + \left(\mathbf{d}^T \cdot \mathbf{B} \cdot \mathbf{y}\right) + c = \left(\mathbf{z}^T \cdot \left(\mathbf{B}^T \cdot \mathbf{A} \cdot \mathbf{B}\right) \cdot \mathbf{z}\right) + \left(c - \tfrac{1}{4}\mathbf{d}^T \cdot \mathbf{A}^{-1} \cdot \mathbf{d}\right) = \left(\mathbf{z}^T \cdot \left(\mathbf{B}^T \cdot \mathbf{A} \cdot \mathbf{B}\right) \cdot \mathbf{z}\right) + \frac{|\mathbf{Q}|}{|\mathbf{A}|}$$

$$\text{mit } a_{ij} = \mathbf{a}_i \cdot \mathbf{a}_j \qquad \mathbf{Q} = \begin{pmatrix} \mathbf{A} & \frac{1}{2}\mathbf{d} \\ \frac{1}{2}\mathbf{d}^T & c \end{pmatrix}$$

Hauptachsentransformation	Translation/quadratische Ergänzung
$\mathbf{x} = \mathbf{B} \cdot \mathbf{y}$	$z_i = y_i + \frac{\mathbf{d}^T \cdot \mathbf{b}_i}{2\lambda_i}$

Polynom

$$P_n(x) = \sum_{i=0}^{n} a_i x^i = a_n \prod_{i=1}^{n} (x - x_i)^{\alpha_i}$$

Partialbruchzerlegung

$$Q_m(x) = a_m \prod_{i=1}^{r} (x - x_i)^{\alpha_i} \prod_{j=1}^{s} \left(x^2 + p_j x + q_j\right)^{\beta_j} \qquad \sum_{i=1}^{r} \alpha_i + \sum_{j=1}^{s} \beta_j = m$$

$$\frac{P_n(x)}{Q_m(x)} = \sum_{i=1}^{r} \sum_{j=1}^{r} \frac{A_{i\alpha_j}}{(x-x_i)^{\alpha_j}} + \sum_{i=1}^{s} \sum_{j=1}^{s} \frac{M_{i\beta_j} x + N_{i\beta_j}}{(x^2+p_i x+q_i)^{\beta_j}}$$

Reihen

$$I = \int_0^\infty f(\mathbf{x}, n)\mathrm{d}n \neq \infty \quad \Leftrightarrow \quad \sum_{n=0}^{\infty} f(\mathbf{x}, n) \neq \infty$$

Konvergenz

$$\lim_{n\to\infty} (f(n)) \neq 0 \quad \Rightarrow \quad I = \infty$$

$$\lim_{n\to\infty} (n\, f(n)) \neq 0 \quad \Rightarrow \quad I = \infty$$

$$\lim_{n\to\infty} \left(\left|\frac{f(n+1)}{f(n)}\right|\right) > 1 \quad \Rightarrow \quad I = \infty$$

$$\lim_{n\to\infty} \left(\sqrt[n]{f(n)}\right) > 1 \quad \Rightarrow \quad I = \infty$$

$$\lim_{n\to\infty} \left(\frac{f(n)}{g(n)}\right) \notin \{0, \infty\} \quad \wedge \quad \int_0^\infty g(n)\mathrm{d}n \neq \infty \quad \Rightarrow \quad I \neq \infty$$

$$\lim_{n\to\infty} \left(\left|\frac{f(n+1)}{f(n)}\right|\right) < 1 \quad \Rightarrow \quad I \neq \infty$$

$$\lim_{n\to\infty} \left(\sqrt[n]{f(n)}\right) < 1 \quad \Rightarrow \quad I \neq \infty$$

$$\int_0^\infty |f(n)|\, \mathrm{d}n \neq \infty \quad \Rightarrow \quad I \neq \infty$$

$$f(n) = (-1)^n\, |g(n)| \quad \wedge \quad \lim_{n\to\infty} (g(n)) = 0 \quad \Rightarrow \quad I \neq \infty$$

$$\int_N^\infty f(n)\mathrm{d}n \leq \sum_{n=N}^{\infty} f(n) = S - \sum_{n=1}^{N-1} f(n) \leq f(N) + \int_N^\infty f(n)\mathrm{d}n$$

Geometrische Reihe

$$s_n = a \sum_{i=0}^{n-1} q^i = a \frac{1-q^n}{1-q}$$

Taylorreihe

$$T(x) = \sum_{i=0}^{n} \frac{f^{(i)}(x_0)}{i!} (x - x_0)^i + R_n(x) \qquad R_n(x) = \int_{x_0}^{x} \frac{(x-t)^n}{n!} f^{(n+1)}(t)\mathrm{d}t = \frac{f^{(i+1)}(x \ldots x_0)}{(i+1)!} (x - x_0)^{i+1}$$

$$T(\mathbf{x} + \mathrm{d}\mathbf{x}) = \sum_{k=0}^{n} \frac{\mathrm{d}^k f(\mathbf{x})}{k!} + R_n(\mathbf{x}) \qquad R_n(\mathbf{x} + \mathrm{d}\mathbf{x}) = \frac{\mathrm{d}^{k+1} f(\mathbf{x} \ldots (\mathbf{x}+\mathrm{d}\mathbf{x}))}{(k+1)!}$$